Project 2.4a: The Precise Definition of a Limit

Objective
To investigate the precise definition of limit.

Narrative
If you have not already done so, read Section 2.4 of the text.

To prove that the number \( L \) which we guess to be the limit of \( f(x) \) at \( x = a \) really is the limit of \( f(x) \) at \( x = a \), we must verify the condition in the formal definition of limit. This condition requires that for each real number \( \epsilon > 0 \), there is a real number \( \delta > 0 \) such that the values of \( f(x) \) for all \( x \) in the interval \((a - \delta, a + \delta)\) — except possibly at \( x = a \) itself — lie between \( L - \epsilon \) and \( L + \epsilon \). In this project we investigate the graphical implications of this condition.

In this project we also introduce the command \texttt{with(<package name>)} which loads the package of routines named <package name> into Maple. We also illustrate how to draw a line segment from the point \( P(a, b) \) to the point \( Q(c, d) \) using the command \texttt{<segment name> := [[a,b],[c,d]]}, and the use of plotting options such as \texttt{color=blue} and \texttt{scaling=constrained}. Finally, we illustrate “delayed plotting” and the use of the \texttt{plots} command \texttt{display}.

Task
a) Type the command lines below into Maple in the order in which they are listed. These commands are concerned with \( \lim_{x \to 2} (-x^3/12 + x^2/2 + 5/3) \). Note that in response to the command “\texttt{> with(plots):}”, Maple will issue the warning “\texttt{Warning, the name changecoords has been redefined}”; warnings such as this are perfectly natural. (On the other hand, as we said before, error messages such as “\texttt{Error ...}” are not! A message such as this is telling you that a mistake has been made that must be corrected before going on!) Also note that we terminate the \texttt{plot0 := ...} and \texttt{plot1 := ...} lines with a colon “;” rather than a semicolon “;", suppressing the immediate display of these plot structures; later we display them using the \texttt{plots} command \texttt{display}.

\begin{verbatim}
> # Your name, today’s date
> # Project 2.4a: The Precise Definition of a Limit
> restart;
> with(plots):
> f := x -> -x^3/12+x^2/2+5/3;
> a := 2.0;
> L := limit(f(x),x=a);
> xeqa := [[a,0],[a,4]];
> plot0 := plot({f(x),L,xeqa},x=-1..4,y=0..4,color=blue,scaling=constrained):
> display(plot0);
> e := 0.5;
> plot1 := plot({L-e,L+e},x=-1..4,y=0..4,color=red,scaling=constrained):
> display({plot0,plot1});
\end{verbatim}

Maple will sometimes use different scales on the \textit{x}- and \textit{y}-coordinate axes to improve the appearance of a graphic. The effect of the \texttt{scaling=constrained} option is to over-ride this feature: \texttt{scaling=constrained} forces Maple to use the same units on the \textit{x}- and \textit{y}-axes.
b) Continue by typing the command lines below into Maple in the order in which they are listed.

```maple
> e := 0.2;
> plot1 := plot({L-e,L+e},x=-1..4,y=0..4,color=red,scaling=constrained):
> display({plot0,plot1});
```

At this point, make a hard-copy of your typed input and Maple’s responses. Then, ...

c) Label the graphs of \( y = f(x) \), \( y = L \), \( y = L \pm e \), and \( x = a \) on the second graphic you produced in part (a) by hand. Estimate by eye and state a value of \( d \) for which the values of \( f(x) \) for all \( x \) in the interval \( a-d..a+d \) — except possibly at \( x = a \) — lie between \( L-e \) and \( L+e \) when \( e = 0.5 \). Draw the lines whose equations are \( x = a+d \) and \( x = a-d \) by hand on the second graphic you drew in part (a).

d) Label the graphs of \( y = f(x) \), \( y = L \), \( y = L \pm e \), and \( x = a \) on the graphic you produced in part (b) by hand. Estimate by eye and state a value of \( d \) for which the values of \( f(x) \) for all \( x \) in the interval \( a-d..a+d \) — except possibly at \( x = a \) — lie between \( L-e \) and \( L+e \) when \( e = 0.2 \). Draw the lines whose equations are \( x = a+d \) and \( x = a-d \) by hand on this graphic.

**Comments**

1. In this project we are not actually proving that \( L = \lim_{x \to a} f(x) \). On one hand, we are just verifying that an appropriate \( d \) exists for two given \( e \)'s: to verify that \( L = \lim_{x \to a} f(x) \), we would have to do this for every \( e \), not just two, three, four, or any finite number of \( e \)'s. On the other hand, since Maple draws the graphs of functions by “connecting-the-dots”, some significant behavior could occur between the dots that is not revealed by Maple, so we cannot trust Maple’s graphics to be completely accurate. This is one of the big reasons the \( \epsilon \delta \)-analysis of limits is so important.

2. You have to tell Maple you want to use a package of routines, such as plots, by saying `with(plots)` only once at the beginning (or right after the `restart`) of a session — not every time you use a routine in the package.

3. If the same options are to be used several times in a Maple session, they can be specified once at the beginning of the session using the `setoptions` command and omitted thereafter (saving time and reducing typing). For example, if we had included the command

```maple
> setoptions(color=blue,scaling=constrained);
```

immediately after the `with(plots)` command, then in all subsequent plots we would have automatically declared `color=blue` and `scaling=constrained`. 